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CHAOS IN THE $Z(2)$ GAUGE MODEL ON A GENERALIZED BETHE LATTICE OF PLAQUETTES

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Abstract

We investigate the $Z(2)$ gauge model on a generalized Bethe lattice with three plaquette representation of the action. We obtain the cascade of phase transitions according to Feigenbaum scheme leading to chaotic states for some values of parameters of the model. The duality between this gauge model and three site Ising spin model on Husimi tree is shown. The Lyapunov exponent as a new order parameter for the characterization of the model in the chaotic region is considered. The line of the second order phase transition, which corresponds to the points of the first period doubling bifurcation, is also obtained.

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The thermodynamics of the phases and the order of the transitions plays an important role in many branches of physics. It is widthly recognized that a study of QCD phase structure will help to understand the one of the main problems in strong interactions, which is the color confinement (see [1] and references therein). At sufficiently high temperatures ($T_c \sim 200MeV$ ($\sim 2.32 \times 10^{12}K$)) one expects a phase transition in gluodynamics: a color confinement phase (hadronic phase) changes to a deconfinement phase (plasma phase). The question of the order of this finite-temperature transition with physical values of three quark masses is still open. The issue of the transition from a confinement phase at low temperatures to the deconfinement phase at high temperatures can be related to the issue of whether the pure glue vacuum of QCD is $Z(N)$ invariant like the action. As it turns out, the transition may be explained as a spontaneous breaking of the extra $Z(N)$ symmetry at finite temperatures.

On the other hand, it is well known that the vacuum of the non-Abelian gauge theories is non integrable in the classical limit and exhibits dynamical chaos [2, 3, 4] (see also [5] for details and extended literature). The connection between the maximum Lyapunov exponent and the gluon dumping rate in a hot perturbative QCD has been obtained in Ref. [6]. These chaotic solutions are playing an essential role in the self thermalization of the hot quark-gluon plasma in heavy ion collision[5]. It is important to mention also the Ref.[7], where it was shown that the deconfinement phase domains in $SU(2)$ lattice gauge theory have a non-integral dimension near the phase transition.

In this letter we are presenting arguments which leads to chaotic states in the $Z(N)$ gauge theory on the hierarchical lattice. The expectation value of the Polyakov loop operator of the $Z(N)$ theory determines the phase structure of the $SU(N)$ gauge theory at a finite temperature[8, 9]. In particular the restoration of the $SU(2)$ gauge symmetry in standard model and deconfining transition in QCD at a finite temperature are consequence of phase transitions in $Z(2)$ and $Z(3)$ models respectively. Using the dynamical systems or recursive approaches one can obtain an order parameter for the $Z(N)$ gauge model on a hierarchical lattices. In many cases recursive sequence converges to a fixed point and one can obtain a qualitatively more correct phase structure than in conventional mean field approximations [10]. It is necessary to mention that for various reasons it is interesting to generalize lattice gauge actions by including larger interaction loops. For instance, the enlarged gauge action involving new double plaquette interaction terms were proposed and studied in $3d$ and $4d$ [11, 12] and there were obtained qualitative changes in phase diagrams. The $2d$ version of one of these lattice gauge models with $Z(2)$ gauge symmetry formulated on the planar rectangular plaquettes was reduced to the usual spin-1/2 Ising model on the square lattice and the point of the second order phase transition was found [13]. Recently, the Z(3) gauge model with double plaquette representation on the flat (triangular and square) [14] and on a generalized recursive lattice [15] is considered and reduced it to the spin-1 Blume-Emery-Griffiths model [16].

In this letter we construct a $Z(2)$ gauge model with three plaquette representation of the action and the cascade of phase transitions according to Feigenbaum scheme leading to chaotic states for some values of parameters of the model. Then the duality between this gauge model on generalized Bethe lattice of plaquettes and three-site Ising spin model

on Husimi tree [17] becomes obvious.

The generalized Bethe lattice is constructed by successive building up of shells. As a zero shell we take the central plaquette and all subsequent shells come out by gluing up two new plaquettes to each link of a previous shell. As a result we get an infinite dimensional lattice on which three plaquettes are gluing up to each link (Fig.1a). On a lattice the gluon field is described by matrixes $U_{x,\mu}$ which are assigned to the links of the lattice. The $U_{x,\mu}$ are elements of the gauge group itself and $U_{x,\mu} = exp(iag_0 A_\mu(x))$, where g_0 is the unnormalized charge, a - the distance between neighboring sites. In the $Z(2)$ gauge model field variables U_{ij} defined on the links take their values among the group of the two roots of unity $U_{ij} \in \{\pm 1\}$. Then the gauge invariant action in the presence of three plaquette interaction can be written in the form

$$S = -\beta_3 \sum_{3p} U_{3p} - \beta_1 \sum_{p} U_p, \tag{1}$$

where

$$U_p = U_{ij} U_{jk} U_{kl} U_{li}$$

is the product of the gauge variables along the plaquette contour and

$$U_{3p} = U_{p_n} U^{'}_{p_{n+1}} U^{''}_{p_{n+1}} = U_{ij} U_{jk} U_{kl} U_{li} U_{im} U_{mn} U_{nj} U_{jq} U_{qr} U_{ri}$$

is the minimal product of the gauge variable along the tree plaquettes (Fig. 2b). The β_1 and β_3 are the gauge coupling constants.

The partition function of this model is

$$Z = \sum_{\{U\}} e^{-S}, \tag{2}$$

where the sum is over all possible configurations of the gauge field variables $\{U\}$. The expectation value of the central single plaquette P will have the following form

$$P \equiv < U_{p_0} > = Z^{-1} \sum_{\{U\}} U_{p_0} e^{-S}. \tag{3}$$

The advantage of the generalized Bethe lattice is that for the models formulated on it exact recursion relation can be derived. The partition function separates into four identical branches, when we are cutting apart the zero shell (central plaquette) of the generalized Bethe lattice. Then the partition function for nth generation ($n \to \infty$ corresponds to the thermodynamic limit) can be rewritten

$$Z_n = \sum_{\{U_{p_0}\}} e^{\beta_1 U_{p_0}} g_n^4(U_{p_0}), \tag{4}$$

where the sum is over all possible configurations of the field variables defined on the links of a zero plaquette $\{U_{p_0}\}$ and

$$g_n(U_{pi}) = \sum_{\{U^{'}_{p_{i+1}}, U^{''}_{p_{i+1}}\}} e^{\beta_3 U_{p_i} U^{'}_{p_{i+1}} U^{''}_{p_{i+1}} + \beta_1 U^{'}_{p_{i+1}} + \beta_1 U^{''}_{p_{i+1}}} g_{n-1}^3(U^{'}_{p_{i+1}}) g_{n-1}^3(U^{''}_{p_{i+1}}). \tag{5}$$

From Eq. (5) one can obtain:

$$g_n(+) = 16e^{\beta_3+2\beta_1} g_{n-1}^3(+)g_{n-1}^3(+) + 32e^{-\beta_3} g_{n-1}^3(+)g_{n-1}^3(-) + 16e^{\beta_3-2\beta_1} g_{n-1}^3(-)g_{n-1}^3(-),$$

$$g_n(-) = 16e^{-\beta_3+2\beta_1} g_{n-1}^3(+)g_{n-1}^3(+) + 32e^{\beta_3} g_{n-1}^3(+)g_{n-1}^3(-) + 16e^{-\beta_3-2\beta_1} g_{n-1}^3(-)g_{n-1}^3(-).$$

Note that U_{p_i} takes the values ± 1 and coefficients before exponents arise because of the gauge invariance.

After introducing the variable

$$x_n = \frac{g_n(+)}{g_n(-)}. \tag{6}$$

one can obtain the following recursion relation

$$x_n = f(x_{n-1}), \qquad f(x) = \frac{z\mu^2 x^6 + 2\mu x^3 + z}{\mu^2 x^6 + 2z\mu x^3 + 1}, \tag{7}$$

where $z = e^{2\beta_3}$, $\quad \mu = e^{2\beta_1}$.

Through this x_n one can express the average value of the central plaquette for nth generation

$$P_n = \frac{8\{e^{\beta_1} g_n^4(+) - e^{-\beta_1} g_n^4(-)\}}{8\{e^{\beta_1} g_n^4(+) + e^{-\beta_1} g_n^4(-)\}} = \frac{e^{\beta_1} x_n^4 - 1}{e^{\beta_1} x_n^4 + 1}, \tag{8}$$

which is the gauge invariant order parameter in a $Z(2)$ theory for a stable fixed point in thermodynamic limit ($n \to \infty$).

Note that by using duality relation the above recursive relation can be obtained for the magnetization m of the three site interacting Ising spin model on Husimi tree with Hamiltonian[17].

$$-\beta H = J_3 \sum_{\triangle} \sigma_i \sigma_j \sigma_k + h \sum_i \sigma_i, \tag{9}$$

where σ_i takes values ± 1, the first sum goes over all triangular faces of the Husimi tree and the second over all sites.

So, we find a relation between $Z(2)$ gauge model with three plaquette representation of action on a generalized Bethe lattice and the three site interacting Ising spin model on Husimi tree. The duality brings to the following correspondence:

$$\begin{array}{cc} U_{p_i} \Leftrightarrow \sigma_i & P \Leftrightarrow m \\ U_{p_i} U_{p_j} U_{p_k} \Leftrightarrow \sigma_i \sigma_j \sigma_k & \beta_1, \beta_3 \Leftrightarrow h, J_3 \end{array}$$

This duality becomes obvious if one construct the dual lattice to the generalized Bethe lattice of plaquettes by joining nearest centers of plaquettes to each other. Indeed, as a result we get the Husimi tree (Fig. 2).

The plot of the P for different values of β_1, β_3 ($\beta_3 = -1/kT$) is presented in Fig 3 in the thermodynamic limit. For high β_3 one has a stable fixed point for P. Decreasing β_3 one can receive a periodic orbit of period 2 for P. Further decreasing β_3 one can get a periodic orbit of period 2^n or a chaotic attractor which does not have a stable

periodic orbits. The last one (strange attractor) we shall call the chaotic state of the $Z(2)$ gauge theory which has a rich phase structure. Note that at high and low values of β_1, P completely determines the state of the system for a stable fixed point but for some intermediate values of β_1 (corresponded to the chaotic region) we can't consider P as an order parameter. The reason of such phase transitions is the different geometrical and dynamical properties of the attracting sets[18] of the map Eq.(7) at different values of β_1 and β_3. The relevant information about the geometrical and dynamical properties of attractors can be found from the generalized dimensions [19] or Lyapunov exponent. To be a good order parameter, a quantity must be capable of describing and characterizing qualitative changes at the bifurcation points [20]. Thus for characterization of the $Z(2)$ gauge model on the generalized Bethe lattice in the chaotic region we have to consider the generalized dimensions or Lyapunov exponents as an order parameter. Recently, we have calculated the Lyapunov exponent of map Eq.(7) in case of the fully developed chaos and shown its nonanalitic behavior[21].

Note that the phase structure of $Z(2)$ model on generalized Bethe lattice allow one to take continuum limit at rich sets of β_1, β_3 of second order phase transitions. In particular, we solve numerically the following system of equations

$$\begin{cases} f(x) - x = 0 \\ f^{'}(x) = -1 \end{cases},$$

which determines the points of the first period doubling bifurcation, i.e. the points of the second order phase transition from the disordered to the two-sublattice ordered phase. The result is presented in Fig.4.

In summary we have shown that $Z(2)$ model on generalized Bethe lattice exhibits cascade of phase transition according to Feigenbaum scheme in the presence of three plaquette interaction. The duality between this gauge model and three site Ising spin model on Husimi tree is shown. We obtained chaotic states at some values of parameter at which order parameter exhibits chaotic behavior. These initial results have opened new challenges for theories of stochastic processes, especially in the direction of stochasticity of vacuum in QCD. We hope that the investigation of the $Z(3)$ gauge theory in the same approach will give a possibility to find out information about confinement-deconfinement phase transition and quark-gluon plasma formation [5].

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Figure Captions

Fig.1. a - Generalized Bethe lattice of plaquettes. b - The minimal product of the gauge variable along the three plaquettes. The arrows show the bypass routing of loop.

Fig.2. The duality between the generalized Bethe lattice of plaquettes and the three site Husimi tree. The sites are denoted by circles.

Fig.3. Plots of P versus β_1 for different temperatures T ($\beta_3 = -1/kT$). a - $T = 2$, b - $T = 1$, c - $T = 0.6$, d - $T = 0.3$.

Fig.4. The line of the second order phase transitions from the disordered to the two-sublattice ordered phase corresponded to the first points of the period doubling bifurcation.

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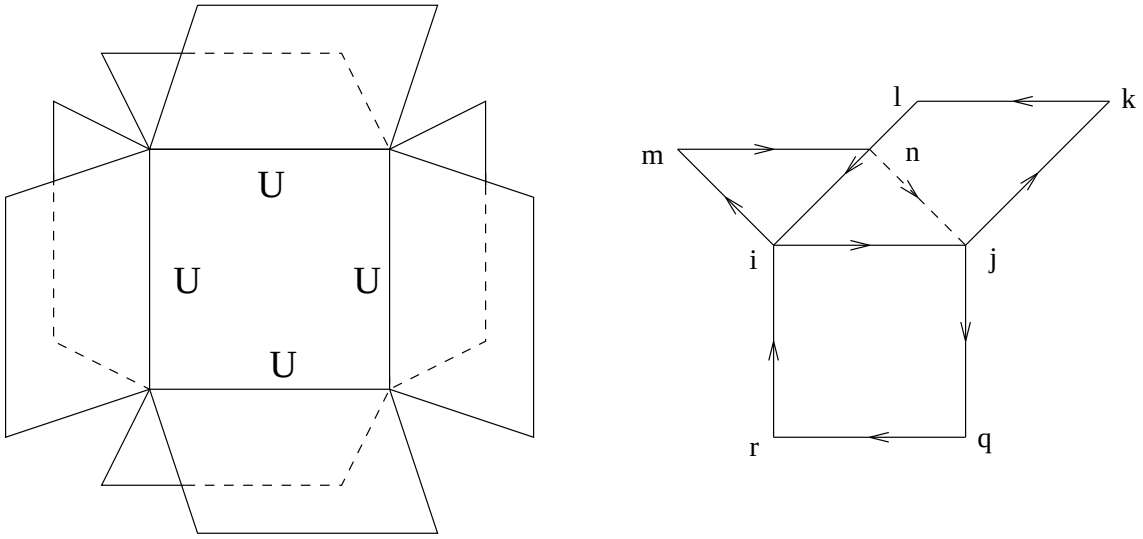


a b

Fig. 1

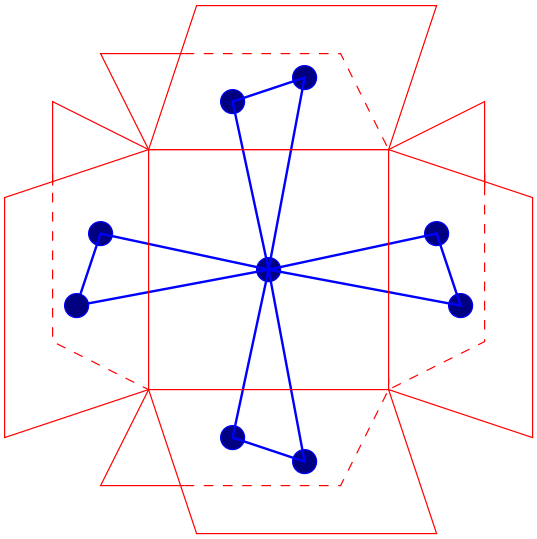
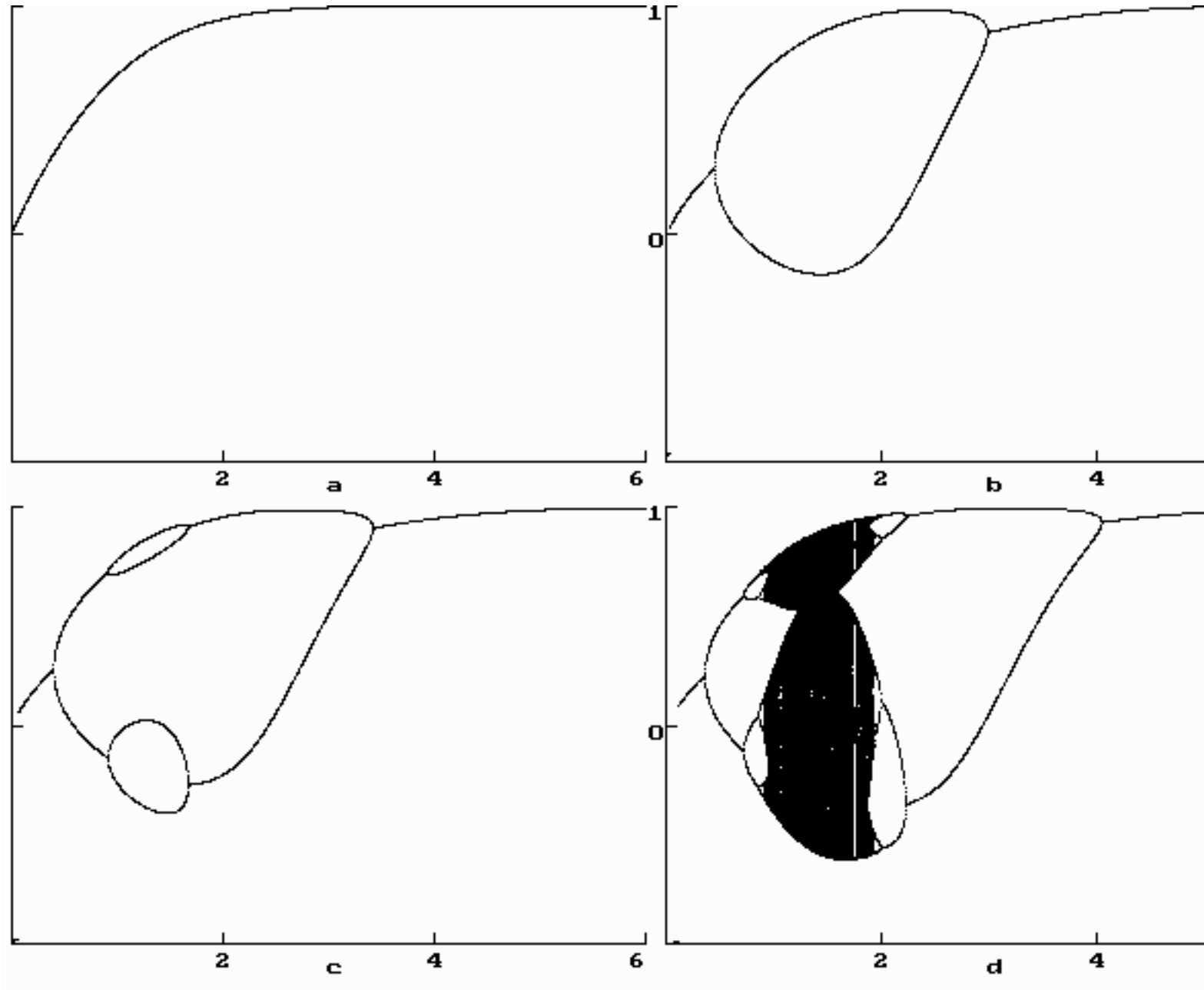


Fig. 2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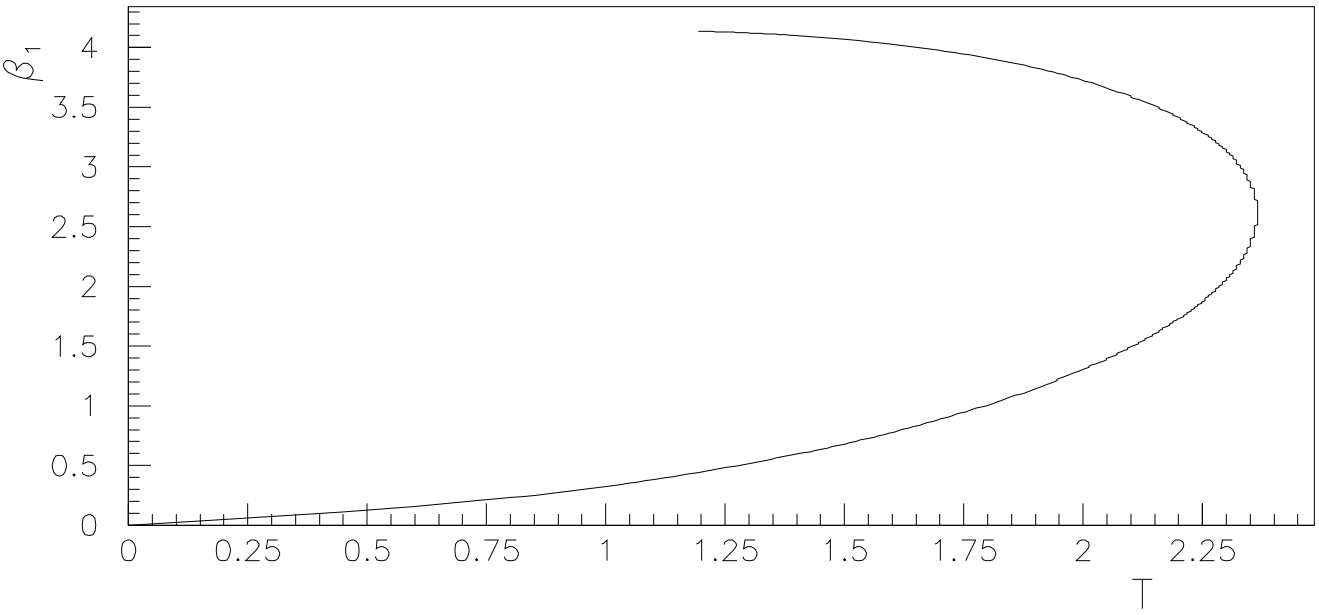

Fig.4